A sequence can be thought of as a list of numbers written in a definite order:

\[ a_1, a_2, a_3, a_4, \ldots, a_n, \ldots \]

The number \( a_1 \) is called the first term, \( a_2 \) is the second term, and in general \( a_n \) is the \( n \)th term. We will deal exclusively with infinite sequences and so each term \( a_n \) will have a successor \( a_{n+1} \).

Notice that for every positive integer \( n \) there is a corresponding number and so a sequence can be defined as a function whose domain is the set of positive integers. But we usually write \( a_n \) instead of the function notation \( f(n) \) for the value of the function at the number \( n \).

**Notation** The sequence \( \{a_1, a_2, a_3, \ldots \} \) is also denoted by

\[ \{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty} \]

**Example 1** Some sequences can be defined by giving a formula for the \( n \)th term. In the following examples we give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula, and a third by writing out the terms of the sequence. Notice that \( n \) doesn’t have to start at 1.

(a) \[ \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} \quad \text{with} \quad a_n = \frac{n}{n+1} \quad \{1, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\} \]

(b) \[ \left\{ \frac{(-1)^n(n+1)}{3^n} \right\}_{n=1}^{\infty} \quad \text{with} \quad a_n = \frac{(-1)^n(n+1)}{3^n} \quad \left\{ \frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \frac{5}{81}, \ldots, \frac{(-1)^n(n+1)}{3^n}, \ldots \right\} \]

(c) \[ \left\{ \sqrt{n-3} \right\}_{n=3}^{\infty} \quad \text{with} \quad a_n = \sqrt{n-3}, \quad n \geq 3 \quad \{0, 1, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots \} \]

(d) \[ \left\{ \cos \frac{n\pi}{6} \right\}_{n=0}^{\infty} \quad \text{with} \quad a_n = \cos \frac{n\pi}{6}, \quad n \geq 0 \quad \left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots, \cos \frac{n\pi}{6}, \ldots \right\} \]

**Example 2** Find a formula for the general term \( a_n \) of the sequence

\[ \left\{ \frac{3}{5'}, \frac{4}{25'}, \frac{5}{125'}, \frac{6}{625'}, \frac{7}{3125'}, \ldots \right\} \]

assuming that the pattern of the first few terms continues.

**Solution** We are given that

\[ a_1 = \frac{3}{5}, \quad a_2 = -\frac{4}{25}, \quad a_3 = \frac{5}{125}, \quad a_4 = -\frac{6}{625}, \quad a_5 = \frac{7}{3125} \]

Notice that the numerators of these fractions start with 3 and increase by 1 whenever we go to the next term. The second term has numerator 4, the third term has numerator 5; in general, the \( n \)th term will have numerator \( n + 2 \). The denominators are the powers of 5, so \( a_n \) has denominator \( 5^n \). The signs of the terms are alternately positive and negative, so
we need to multiply by a power of \(-1\). In Example 1(b) the factor \((-1)^n\) meant we started with a negative term. Here we want to start with a positive term and so we use \((-1)^{n-1}\) or \((-1)^n\). Therefore

\[
a_n = (-1)^{n-1} \cdot \frac{n + 2}{5^n}
\]

**EXAMPLE 3** Here are some sequences that don’t have a simple defining equation.

(a) The sequence \(\{p_n\}\), where \(p_n\) is the population of the world as of January 1 in the year \(n\).

(b) If we let \(a_n\) be the digit in the \(n\)th decimal place of the number \(e\), then \(\{a_n\}\) is a well-defined sequence whose first few terms are

\[
\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \ldots\}
\]

(c) The **Fibonacci sequence** \(\{f_n\}\) is defined recursively by the conditions

\[
f_1 = 1 \quad f_2 = 1 \quad f_n = f_{n-1} + f_{n-2} \quad n \geq 3
\]

Each term is the sum of the two preceding terms. The first few terms are

\[
\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}
\]

This sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits (see Exercise 71).

A sequence such as the one in Example 1(a), \(a_n = n/(n + 1)\), can be pictured either by plotting its terms on a number line as in Figure 1 or by plotting its graph as in Figure 2. Note that, since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated points with coordinates

\[
(1, a_1) \quad (2, a_2) \quad (3, a_3) \quad \ldots \quad (n, a_n) \quad \ldots
\]

From Figure 1 or 2 it appears that the terms of the sequence \(a_n = n/(n + 1)\) are approaching 1 as \(n\) becomes large. In fact, the difference

\[
1 - \frac{n}{n + 1} = \frac{1}{n + 1}
\]

can be made as small as we like by taking \(n\) sufficiently large. We indicate this by writing

\[
\lim_{n \to \infty} \frac{n}{n + 1} = 1
\]

In general, the notation

\[
\lim_{n \to \infty} a_n = L
\]

means that the terms of the sequence \(\{a_n\}\) approach \(L\) as \(n\) becomes large. Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity given in Section 2.6.
A sequence has the limit $L$ and we write
$$\lim_{n \to \infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty$$
if we can make the terms $a_n$ as close to $L$ as we like by taking $n$ sufficiently large. If $\lim_{n \to \infty} a_n$ exists, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

Figure 3 illustrates Definition 1 by showing the graphs of two sequences that have the limit $L$.

![Figure 3](image)

Graphs of two sequences with $\lim_{n \to \infty} a_n = L$

A more precise version of Definition 1 is as follows.

Definition 2 is illustrated by Figure 4, in which the terms $a_1, a_2, a_3, \ldots$ are plotted on a number line. No matter how small an interval $(L - \varepsilon, L + \varepsilon)$ is chosen, there exists an $N$ such that all terms of the sequence from $a_{N+1}$ onward must lie in that interval.

![Figure 4](image)

Another illustration of Definition 2 is given in Figure 5. The points on the graph of $\{a_n\}$ must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small $\varepsilon$ is chosen, but usually a smaller $\varepsilon$ requires a larger $N$.

![Figure 5](image)
If you compare Definition 2 with Definition 2.6.7, you will see that the only difference between \( \lim_{n \to \infty} a_n = L \) and \( \lim_{x \to \infty} f(x) = L \) is that \( n \) is required to be an integer. Thus we have the following theorem, which is illustrated by Figure 6.

**THEOREM**  If \( \lim_{x \to \infty} f(x) = L \) and \( f(n) = a_n \) when \( n \) is an integer, then  
\[
\lim_{n \to \infty} a_n = L.
\]

In particular, since we know that \( \lim_{n \to \infty} (1/n^r) = 0 \) when \( r > 0 \) (Theorem 2.6.5), we have  
\[
\lim_{n \to \infty} \frac{1}{n^r} = 0 \quad \text{if} \quad r > 0.
\]

If \( a_n \) becomes large as \( n \) becomes large, we use the notation \( \lim_{n \to \infty} a_n = \infty \). The following precise definition is similar to Definition 2.6.9.

**DEFINITION**  \( \lim_{n \to \infty} a_n = \infty \) means that for every positive number \( M \) there is an integer \( N \) such that  
\[
\text{if} \quad n > N \quad \text{then} \quad a_n > M.
\]

If \( \lim_{n \to \infty} a_n = \infty \), then the sequence \( \{a_n\} \) is divergent but in a special way. We say that \( \{a_n\} \) diverges to \( \infty \).

The Limit Laws given in Section 2.3 also hold for the limits of sequences and their proofs are similar.

**LIMIT LAWS FOR SEQUENCES**

If \( \{a_n\} \) and \( \{b_n\} \) are convergent sequences and \( c \) is a constant, then  
\[
\begin{align*}
\lim (a_n + b_n) &= \lim a_n + \lim b_n, \\
\lim (a_n - b_n) &= \lim a_n - \lim b_n, \\
\lim ca_n &= c \lim a_n, \\
\lim (a_nb_n) &= \lim a_n \cdot \lim b_n, \\
\lim \frac{a_n}{b_n} &= \frac{\lim a_n}{\lim b_n} \quad \text{if} \quad \lim b_n \neq 0, \\
\lim a_n^p &= \left( \lim a_n \right)^p \quad \text{if} \quad p > 0 \text{ and } a_n > 0.
\end{align*}
\]
The Squeeze Theorem can also be adapted for sequences as follows (see Figure 7).

If \( a_n \leq b_n \leq c_n \) for \( n \geq n_0 \) and \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L \), then \( \lim_{n \to \infty} b_n = L \).

Another useful fact about limits of sequences is given by the following theorem, whose proof is left as Exercise 75.

**Theorem**

If \( \lim_{n \to \infty} |a_n| = 0 \), then \( \lim_{n \to \infty} a_n = 0 \).

**Example 4** Find \( \lim_{n \to \infty} \frac{n}{n + 1} \).

**Solution** The method is similar to the one we used in Section 2.6: Divide numerator and denominator by the highest power of \( n \) and then use the Limit Laws.

\[
\lim_{n \to \infty} \frac{n}{n + 1} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} = \frac{\lim_{n \to \infty} 1}{\lim_{n \to \infty} 1 + \lim_{n \to \infty} \frac{1}{n}} = \frac{1}{1 + 0} = 1
\]

Here we used Equation 4 with \( r = 1 \).

**Example 5** Calculate \( \lim_{n \to \infty} \frac{\ln n}{n} \).

**Solution** Notice that both numerator and denominator approach infinity as \( n \to \infty \). We can’t apply l’Hospital’s Rule directly because it applies not to sequences but to functions of a real variable. However, we can apply l’Hospital’s Rule to the related function \( f(x) = (\ln x)/x \) and obtain

\[
\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0
\]

Therefore, by Theorem 3, we have

\[
\lim_{n \to \infty} \frac{\ln n}{n} = 0
\]

**Example 6** Determine whether the sequence \( a_n = (-1)^n \) is convergent or divergent.

**Solution** If we write out the terms of the sequence, we obtain

\[ \{-1, 1, -1, 1, -1, 1, -1, \ldots \} \]

The graph of this sequence is shown in Figure 8. Since the terms oscillate between 1 and \(-1\) infinitely often, \( a_n \) does not approach any number. Thus \( \lim_{n \to \infty} (-1)^n \) does not exist; that is, the sequence \( \{(-1)^n\} \) is divergent.
EXAMPLE 7 Evaluate \( \lim_{n \to \infty} \frac{(-1)^n}{n} \) if it exists.

SOLUTION

\[
\lim_{n \to \infty} \frac{(-1)^n}{n} = \lim_{n \to \infty} \frac{1}{n} = 0
\]

Therefore, by Theorem 6,

\[
\lim_{n \to \infty} \frac{(-1)^n}{n} = 0
\]

The following theorem says that if we apply a continuous function to the terms of a convergent sequence, the result is also convergent. The proof is left as Exercise 76.

**THEOREM**

If \( \lim a_n = L \) and the function \( f \) is continuous at \( L \), then

\[
\lim f(a_n) = f(L)
\]

EXAMPLE 8 Find \( \lim \sin(\pi/n) \).

SOLUTION Because the sine function is continuous at 0, Theorem 7 enables us to write

\[
\lim \sin(\pi/n) = \sin \left( \lim \frac{\pi}{n} \right) = \sin 0 = 0
\]

EXAMPLE 9 Discuss the convergence of the sequence \( a_n = n!/n^n \), where \( n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \).

SOLUTION Both numerator and denominator approach infinity as \( n \to \infty \), but here we have no corresponding function for use with l’Hospital’s Rule (\( x! \) is not defined when \( x \) is not an integer). Let’s write out a few terms to get a feeling for what happens to \( a_n \) as \( n \) gets large:

\[
a_1 = 1, \quad a_2 = \frac{1 \cdot 2}{2 \cdot 2}, \quad a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}
\]

\[
a_n = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{n \cdot n \cdot n \cdot \ldots \cdot n}
\]

It appears from these expressions and the graph in Figure 10 that the terms are decreasing and perhaps approach 0. To confirm this, observe from Equation 8 that

\[
a_n = \frac{1}{n} \left( \frac{2 \cdot 3 \cdot \ldots \cdot n}{n \cdot n \cdot n \cdot \ldots \cdot n} \right)
\]

Notice that the expression in parentheses is at most 1 because the numerator is less than (or equal to) the denominator. So

\[
0 < a_n \leq \frac{1}{n}
\]

We know that \( 1/n \to 0 \) as \( n \to \infty \). Therefore \( a_n \to 0 \) as \( n \to \infty \) by the Squeeze Theorem.
EXAMPLE 10  For what values of $r$ is the sequence $\{r^n\}$ convergent?

SOLUTION  We know from Section 2.6 and the graphs of the exponential functions in Section 1.5 that $\lim_{x \to \infty} a^x = \infty$ for $a > 1$ and $\lim_{x \to \infty} a^x = 0$ for $0 < a < 1$. Therefore, putting $a = r$ and using Theorem 3, we have

$$\lim_{n \to \infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$$

It is obvious that

$$\lim_{n \to \infty} 1^n = 1 \quad \text{and} \quad \lim_{n \to \infty} 0^n = 0$$

If $-1 < r < 0$, then $0 < |r| < 1$, so

$$\lim_{n \to \infty} |r^n| = \lim_{n \to \infty} |r|^n = 0$$

and therefore $\lim_{n \to \infty} r^n = 0$ by Theorem 6. If $r \leq -1$, then $\{r^n\}$ diverges as in Example 6. Figure 11 shows the graphs for various values of $r$. (The case $r = -1$ is shown in Figure 8.)

The results of Example 10 are summarized for future use as follows.

9. The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of $r$.

$$\lim_{n \to \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

10. DEFINITION  A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. It is called monotonic if it is either increasing or decreasing.

EXAMPLE 11  The sequence $\left\{ \frac{3}{n+5} \right\}$ is decreasing because

$$\frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

and so $a_n > a_{n+1}$ for all $n \geq 1$. 

\[\]
EXAMPLE 12  Show that the sequence \( a_n = \frac{n}{n^2 + 1} \) is decreasing.

SOLUTION 1  We must show that \( a_{n+1} < a_n \), that is,

\[
\frac{n + 1}{(n + 1)^2 + 1} < \frac{n}{n^2 + 1}
\]

This inequality is equivalent to the one we get by cross-multiplication:

\[
\frac{n + 1}{(n + 1)^2 + 1} < \frac{n}{n^2 + 1} \iff (n + 1)(n^2 + 1) < n[(n + 1)^2 + 1] \\
\iff n^3 + 2n + n + 1 < n^3 + 2n^2 + 2n \\
\iff 1 < n^2 + n
\]

Since \( n \geq 1 \), we know that the inequality \( n^2 + n > 1 \) is true. Therefore \( a_{n+1} < a_n \) and so \( \{a_n\} \) is decreasing.

SOLUTION 2  Consider the function \( f(x) = \frac{x}{x^2 + 1} \):

\[
f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} < 0 \quad \text{whenever } x^2 > 1
\]

Thus \( f \) is decreasing on \((1, \infty)\) and so \( f(n) > f(n + 1) \). Therefore \( \{a_n\} \) is decreasing.

DEFINITION  A sequence \( \{a_n\} \) is bounded above if there is a number \( M \) such that

\[
a_n \leq M \quad \text{for all } n \geq 1
\]

It is bounded below if there is a number \( m \) such that

\[
m \leq a_n \quad \text{for all } n \geq 1
\]

If it is bounded above and below, then \( \{a_n\} \) is a bounded sequence.

For instance, the sequence \( a_n = n \) is bounded below \((a_n > 0)\) but not above. The sequence \( a_n = n/(n + 1) \) is bounded because \( 0 < a_n < 1 \) for all \( n \).

We know that not every bounded sequence is convergent [for instance, the sequence \( a_n = (-1)^n \) satisfies \(-1 \leq a_n \leq 1 \) but is divergent from Example 6] and not every monotonic sequence is convergent \((a_n = n \to \infty)\). But if a sequence is both bounded and monotonic, then it must be convergent. This fact is proved as Theorem 12, but intuitively you can understand why it is true by looking at Figure 12. If \( \{a_n\} \) is increasing and \( a_n \leq M \) for all \( n \), then the terms are forced to crowd together and approach some number \( L \).

The proof of Theorem 12 is based on the Completeness Axiom for the set \( \mathbb{R} \) of real numbers, which says that if \( S \) is a nonempty set of real numbers that has an upper bound \( M \) \((x \leq M \text{ for all } x \in S)\), then \( S \) has a least upper bound \( b \). (This means that \( b \) is an upper bound for \( S \), but if \( M \) is any other upper bound, then \( b \leq M \).) The Completeness Axiom is an expression of the fact that there is no gap or hole in the real number line.
MONOTONIC SEQUENCE THEOREM  Every bounded, monotonic sequence is convergent.

PROOF  Suppose \( \{a_n\} \) is an increasing sequence. Since \( \{a_n\} \) is bounded, the set \( S = \{a_n | n \geq 1\} \) has an upper bound. By the Completeness Axiom it has a least upper bound \( L \). Given \( \varepsilon > 0 \), \( L - \varepsilon \) is not an upper bound for \( S \) (since \( L \) is the least upper bound). Therefore

\[
a_n > L - \varepsilon \quad \text{for some integer } N
\]

But the sequence is increasing so \( a_n \geq a_n \) for every \( n > N \). Thus if \( n > N \), we have

\[
a_n > L - \varepsilon
\]

so

\[
0 \leq L - a_n < \varepsilon
\]

since \( a_n \leq L \). Thus

\[
|L - a_n| < \varepsilon \quad \text{whenever } n > N
\]

so \( \lim_{n \to \infty} a_n = L \).

A similar proof (using the greatest lower bound) works if \( \{a_n\} \) is decreasing.

The proof of Theorem 12 shows that a sequence that is increasing and bounded above is convergent. (Likewise, a decreasing sequence that is bounded below is convergent.) This fact is used many times in dealing with infinite series.

EXAMPLE 13  Investigate the sequence \( \{a_n\} \) defined by the recurrence relation

\[
a_1 = 2 \quad a_{n+1} = \frac{1}{2}(a_n + 6) \quad \text{for } n = 1, 2, 3, \ldots
\]

SOLUTION  We begin by computing the first several terms:

\[
\begin{align*}
a_1 &= 2 \\
a_2 &= \frac{1}{2}(2 + 6) = 4 \\
a_3 &= \frac{1}{2}(4 + 6) = 5 \\
a_4 &= \frac{1}{2}(5 + 6) = 5.5 \\
a_5 &= 5.75 \\
a_6 &= 5.875 \\
a_7 &= 5.9375 \\
a_8 &= 5.96875 \\
a_9 &= 5.984375
\end{align*}
\]

These initial terms suggest that the sequence is increasing and the terms are approaching 6. To confirm that the sequence is increasing, we use mathematical induction to show that \( a_{n+1} > a_n \) for all \( n \geq 1 \). This is true for \( n = 1 \) because \( a_2 = 4 > a_1 \). If we assume that it is true for \( n = k \), then we have

\[
a_{k+1} > a_k
\]

so

\[
a_{k+1} + 6 > a_k + 6
\]

and

\[
\frac{1}{2}(a_{k+1} + 6) > \frac{1}{2}(a_k + 6)
\]

Thus

\[
a_{k+2} > a_{k+1}
\]

Mathematical induction is often used in dealing with recursive sequences. See page 77 for a discussion of the Principle of Mathematical Induction.
We have deduced that \( a_{n+1} > a_n \) is true for \( n = k + 1 \). Therefore the inequality is true for all \( n \) by induction.

Next we verify that \( \{a_n\} \) is bounded by showing that \( a_n < 6 \) for all \( n \). (Since the sequence is increasing, we already know that it has a lower bound: \( a_n \geq a_1 = 2 \) for all \( n \).) We know that \( a_1 < 6 \), so the assertion is true for \( n = 1 \). Suppose it is true for \( n = k \). Then

\[
a_k < 6
\]

so

\[
a_k + 6 < 12
\]

Thus

\[
a_{k+1} < 6
\]

This shows, by mathematical induction, that \( a_n < 6 \) for all \( n \).

Since the sequence \( \{a_n\} \) is increasing and bounded, Theorem 12 guarantees that it has a limit. The theorem doesn’t tell us what the value of the limit is. But now that we know \( L = \lim_{n \to \infty} a_n \) exists, we can use the recurrence relation to write

\[
\lim a_{n+1} = \lim \frac{1}{2}(a_n + 6) = \frac{1}{2}\left( \lim a_n + 6 \right) = \frac{1}{2}(L + 6)
\]

A proof of this fact is requested in Exercise 58.

Since \( a_n \to L \), it follows that \( a_{n+1} \to L \), too (as \( n \to \infty, n + 1 \to \infty \) too). So we have

\[
L = \frac{1}{2}(L + 6)
\]

Solving this equation for \( L \), we get \( L = 6 \), as predicted.

### 11.1 Exercises

1. (a) What is a sequence?
   (b) What does it mean to say that \( \lim_{n \to \infty} a_n = 8 \)?
   (c) What does it mean to say that \( \lim_{n \to \infty} a_n = \infty \)?

2. (a) What is a convergent sequence? Give two examples.
   (b) What is a divergent sequence? Give two examples.

3–8 List the first five terms of the sequence.

3. \( a_n = 1 - (0.2)^n \)

4. \( a_n = \frac{n + 1}{3n - 1} \)

5. \( a_n = \frac{3(-1)^n}{n!} \)

6. \( \{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)\} \)

7. \( a_1 = 3, \quad a_{n+1} = 2a_n - 1 \)

8. \( a_1 = 4, \quad a_{n+1} = \frac{a_n}{a_n - 1} \)

9–14 Find a formula for the general term \( a_n \) of the sequence, assuming that the pattern of the first few terms continues.

9. \( \{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots \} \)

10. \( \{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots \} \)

11. \( \{2, 7, 12, 17, \ldots \} \)

12. \( \{-\frac{1}{7}, \frac{2}{9}, \frac{3}{11}, \frac{4}{13}, \ldots \} \)

13. \( \{1, -\frac{2}{3}, \frac{4}{5}, -\frac{6}{7}, \ldots \} \)

14. \( \{5, 1, 5, 1, 5, 1, \ldots \} \)

15. List the first six terms of the sequence defined by

\[
a_n = \frac{n}{2n + 1}
\]

Does the sequence appear to have a limit? If so, find it.

16. List the first nine terms of the sequence \( \{\cos(n \pi/3)\} \). Does this sequence appear to have a limit? If so, find it. If not, explain why.

17–46 Determine whether the sequence converges or diverges. If it converges, find the limit.

17. \( a_n = 1 - (0.2)^n \)

18. \( a_n = \frac{n^3}{n^3 + 1} \)